can then write $\rho_1(\mathbf{r})-\rho_{1,0}=\alpha\rho_E$ so that

$$-\sum_{k<k_F}\{\langle\mathbf{k}|PV_{2E}|\mathbf{k}\rangle-\langle\mathbf{k}|P|\mathbf{k}\rangle\langle\mathbf{k}|V_{2E}|\mathbf{k}\rangle\}$$
$$=-\sum_{k<k_F}\sum_{q\neq 0}\langle\mathbf{k}|P|\mathbf{k}+\mathbf{q}\rangle\langle\mathbf{k}+\mathbf{q}|V_{2E}|\mathbf{k}\rangle$$
$$=\alpha\int V_{2E}(\mathbf{r})\rho_E(\mathbf{r})d\mathbf{r}. \quad \text{(A3)}$$

On the other hand, $V_{2E}(\mathbf{r})$ is exactly of the same nature as $\tilde{V}_E$ [see Eq. (3.4a)] and the interaction of $\rho_E(\mathbf{r})$ with $[\tilde{V}_E(\mathbf{r})-V_{2E}(\mathbf{r})]$ is just a self-energy term which could be added to (3.11), but is so small that it can be neglected. We can thus identify (A3) with

$$\alpha\int\rho_E(\mathbf{r})\tilde{V}_E(\mathbf{r})d\mathbf{r}. \quad \text{(A4)}$$

Finally from (A2) and from the definition of ρ_0 given in Sec. III C we find that

$$\alpha=\sum_{k<k_F}\langle\mathbf{k}|P|\mathbf{k}\rangle/\sum_{k<k_F}\langle\mathbf{k}|\mathbf{k}\rangle=[\langle\mathbf{k}|P|\mathbf{k}\rangle]_{\text{av}}. \quad \text{(A5)}$$

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Spin-Wave Interaction in the Itinerant-Electron Model of Ferromagnetism*

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Using a simple itinerant-electron model of ferromagnetism with exchange interaction and Coulomb repulsion between band electrons, spin waves, and their interactions are discussed. In the random-phase approximation, we obtained the diagonal elements of spin-wave interactions which reduce to Dyson's result in the limit of localized electrons for our model. A diagrammatical interpretation of the result is also given. When the band with down spin is partially filled and that with up spin empty, the spin-wave interaction consists of the part arising from the modified exchange interaction, the part due to the electron kinetic energy and the Coulomb repulsion, and the part which involves both of these effects. Among these, the second seems to be rather small. For small values of the wave vectors of the spin waves involved, the spin-wave interaction depends on the wave vectors in the same way as in Dyson's result. The modification on the exchange interaction is such that the short-range part of the original exchange interaction is suppressed, whereas the long-range part remains unaffected. This arises from electron (hole) exchanges, and cancels in the limit of localized electrons.

I. INTRODUCTION

SINCE Dyson's theory[1] on the spin-wave interactions of the Heisenberg spin system appeared, this problem has been a subject of many investigations.[2] However, the experimental test of these theories did not appear until recently. Experiments have been performed on ferromagnetic metals such as[3] Ni and[4] permalloy to determine the temperature dependence of spin-wave frequencies which arise from spin-wave interactions. In the low-temperature region, they obtained the spin-wave frequency which decreases with the temperature as $T^{5/2}$, in agreement with Dyson's result. However, its magnitude, which is proportional to the square of the range of exchange interactions, is too large and requires the range of exchange interactions of about eight times the lattice constant to fit the above mentioned theoretical result. There are other evidences[5] which indicate the existence of long-range exchange interactions. However, it was pointed out that such a long-range exchange interaction is inconsistent with other experimental evidence.[4] It has been suggested that the itinerant character of electrons in these metals may be important.[6]

On the other hand this problem is also of theoretical interest as another example in which interactions among elementary excitations play a major role.[1,2,7] Although the property of elementary excitations in many-body systems has been a subject of numerous investigations, not much work has been done on the problem of interactions among them, which are essential in understanding the temperature dependence of energies of ele-

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mentary excitations as well as the dissipation phenomena involving them.

Thus, it seems worthwhile to study the spin-wave interactions based on the itinerant electron model of ferromagnetism. The main difficulty here is the lack of exact description of a single spin wave in contrast to the case of the Heisenberg model. In this respect the situation is similar to that encountered in ultrasonic attenuation in liquid helium at very low temperatures.[7] Nevertheless, choosing a simple itinerant electron model of ferromagnetism which includes the Heisenberg model as a limiting case, it was possible to obtain a spin wave interaction which is a direct extension of that of Dyson.

Although the expression for spin-wave interactions is somewhat complicated, it is possible to give a fairly simple diagrammatic interpretation of the result. Consequences of the results are not fully examined, but we shall study two new aspects which did not exist in the case of the Heisenberg model.

Throughout the text of this paper, we used the method of normal coordinates which seems to give the clearest presentation of the spin-wave interaction. In the Appendix, we shall discuss the connection of this method with that of the Green's function.

II. DESCRIPTION OF SPIN WAVES

The system chosen in this work is described by the following Hamiltonian:

$$\mathcal{H}=\sum_{\mathbf{k},\sigma}\epsilon(\mathbf{k})a_{\mathbf{k}\sigma}{}^{*}a_{\mathbf{k}\sigma}$$
$$+\frac{1}{2N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\sum_{\sigma_1\sigma_2}J(\mathbf{k}_1-\mathbf{k}_3)a_{\mathbf{k}_1\sigma_1}{}^{*}a_{\mathbf{k}_2\sigma_2}{}^{*}a_{\mathbf{k}_3\sigma_2}a_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3\sigma_1}$$
$$+\frac{U}{N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}_1\uparrow}{}^{*}a_{\mathbf{k}_2\uparrow}a_{\mathbf{k}_3\downarrow}{}^{*}a_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3\downarrow}, \quad (2.1)$$

where $a_{\mathbf{k}\sigma}{}^{*}$ and $a_{\mathbf{k}\sigma}$ are creation and annihilation operators of an electron described by the Bloch function with the wave vector $\mathbf{k}$ and spin σ, respectively. The first term is the kinetic energy. The meaning of other terms becomes clearer if we go from Bloch to Wannier representation, where operators in Wannier representation are written as

$$a_{\mathbf{l}\sigma}=\frac{1}{N^{1/2}}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{l}}a_{\mathbf{k}\sigma}, \quad \text{and complex conjugate},$$

where $\mathbf{l}$ designates a lattice site. Then the second and the third terms of (2.1) are written as

$$-\sum_{\mathbf{lm}}J_{\mathbf{lm}}(\mathbf{S}_{\mathbf{l}}\cdot\mathbf{S}_{\mathbf{m}}+S_{\mathbf{l}}{}^{0}S_{\mathbf{m}}{}^{0})+U\sum_{\mathbf{l}}a_{\mathbf{l}\uparrow}{}^{*}a_{\mathbf{l}\uparrow}a_{\mathbf{l}\downarrow}{}^{*}a_{\mathbf{l}\downarrow}, \quad (2.1')$$

where the spin operators $\mathbf{S}_{\mathbf{l}}$ and $S_{\mathbf{l}}{}^{0}$ are defined by,

$$S_{\mathbf{l}}{}^{+}\equiv S_{\mathbf{l}}{}^{x}+iS_{\mathbf{l}}{}^{y}=a_{\mathbf{l}\uparrow}{}^{*}a_{\mathbf{l}\downarrow}, \quad S_{\mathbf{l}}{}^{-}\equiv S_{\mathbf{l}}{}^{x}-iS_{\mathbf{l}}{}^{y}=a_{\mathbf{l}\downarrow}{}^{*}a_{\mathbf{l}\uparrow}$$
$$S_{\mathbf{l}}{}^{z}\equiv\tfrac{1}{2}(a_{\mathbf{l}\uparrow}{}^{*}a_{\mathbf{l}\uparrow}-a_{\mathbf{l}\downarrow}{}^{*}a_{\mathbf{l}\downarrow}), \quad S_{\mathbf{l}}{}^{0}\equiv\tfrac{1}{2}(a_{\mathbf{l}\uparrow}{}^{*}a_{\mathbf{l}\uparrow}+a_{\mathbf{l}\downarrow}{}^{*}a_{\mathbf{l}\downarrow}) \quad (2.2)$$

and $J_{\mathbf{lm}}$ is the Fourier transform of $J(\mathbf{k})$.

Thus, it is clear that the last term in (2.1) represents the Coulomb correlation of electrons at the same lattice point and the second term represents the interaction of electrons at the different sites which includes the exchange interaction.

For simplicity, we assume that the band is nondegenerate except for spins, or assumed no orbital degeneracy. We also assume that the crystal has a cubic symmetry. A somewhat similar Hamiltonian was considered by Ruijgrok.[8] But it appears that he did not consider the Coulomb repulsion U, which should be very important. However, some of his results are still applicable in our model. This model without the interaction between electrons at different sites was also considered by Kubo and his co-workers.[9] The reason for including this is that our model reduces to the Heisenberg model in a certain limit, thus allowing us to compare our result with that of the latter model which is now well established.[1,2]

On the other hand, one might object to the use of our Hamiltonian from a more fundamental point of view, especially with respect to the exchange interaction. Ruijgrok[8] attempted to derive a Hamiltonian similar to ours, but the situation seems to be rather complicated. In fact, this turns out to be a most difficult problem related to the foundation of ferromagnetism. Instead of dealing with this subtle and controversial problem, we regard the Hamiltonian (2.1) as a useful model of itinerant electron ferromagnetism, just as the Heisenberg Hamiltonian was a model of ferromagnetic insulators. Here we shall concentrate on the method of obtaining spin-wave interactions and a qualitative analysis of the results obtained, which will be useful in analyzing a more realistic Hamiltonian in the future.

The Hamiltonian (2.1) has some convenient properties for our problem. Suppose each site is occupied by a single electron of either spin. Then, in the absence of kinetic energy, (2.1) simply reduces to the Heisenberg Hamiltonian of spin $\frac{1}{2}$ plus constant terms. We choose $J_{\mathbf{lm}}$ such that the ground state is always ferromagnetic. Then in the ground state the electron of each site has the same direction of spin which we choose as the negative z axis. Because of the Pauli principle, one can easily verify that the result of the application of the kinetic energy Hamiltonian to this ground state vanishes. [Here we have made use of the fact that without loss of generality we can choose $\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}=0$]. Thus, as long as

[8] T. W. Ruijgrok, Physica **28**, 877 (1962); D. C. Mattis, Phys. Rev. **132**, 2521 (1963).

[9] T. Izuyama, D. J. Kim, and R. Kubo, J. Phys. Soc. Japan **18**, 1025 (1963). T. Izuyama and R. Kubo, J. Appl. Phys. **35**, 1074 (1964).

one can treat the kinetic energy as a perturbation, its effect upon the ground state vanishes. As already noted by Ruijgork,[8] this situation allows us at least in this simple case to obtain exact description of spin waves and other excited states, which is written as follows:

$$\Psi_{\mathbf{q}}=\sum_{\mathbf{k}} c_{\mathbf{k}+\mathbf{q},\mathbf{k}} a_{\mathbf{k}+\mathbf{q}\uparrow}^{*} a_{\mathbf{k}\downarrow}\phi_0\,, \quad \text{(wave function)} \quad (2.3a)$$

where

$$[-\omega_{\mathbf{q}}+\tilde{u}+\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k})]c_{\mathbf{k}+\mathbf{q},\mathbf{k}} =[\tilde{u}-J(0,\mathbf{q})]\frac{1}{N}\sum_{\mathbf{k}'} c_{\mathbf{k}'+\mathbf{q},\mathbf{k}'} \quad (2.3b)$$

and

$$\frac{1}{N}\sum_{\mathbf{k}}\frac{1}{\tilde{u}+\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k})-\omega_{\mathbf{q}}}=\frac{1}{\tilde{u}-J(0,\mathbf{q})}\,, \quad \text{(eigenvalue equation)} \quad (2.3c)$$

where ω_q is the excitation spectrum with wave vector $\mathbf{q}$ determined by solving (2.3c), the lowest value of which corresponds, for small values of q, to the spin-wave energy, and ϕ_0 denotes the ground state, N the number of lattice sites. We also used the following notations:

$$\epsilon(\mathbf{k},\mathbf{k}')\equiv\epsilon(\mathbf{k})-\epsilon(\mathbf{k}')\,, \quad J(\mathbf{k},\mathbf{k}')\equiv J(\mathbf{k})-J(\mathbf{k}') \quad (2.4)$$

and

$$\tilde{u}\equiv U+J(0)\,.$$

Band-theoretically, the case considered above corresponds to the situation in which in the ground state the band with down spin is completely filled, whereas that with up spin is empty, and electrons are bound to each lattice site except for polarizations in spin waves and other excited states.[10] In the itinerant-electron model, on the other hand, electrons are not bound to the sites even in the ground state, and thus it is essential to start with the ground state in which at least one band is partially filled. Unfortunately, however, the exact description of the ground as well as the excited states for such a situation has not been obtained, and it is inevitable to introduce a certain approximation. We choose an approximation in such a way that it reduces to the exact solution in the above-mentioned exactly soluble case. This enables us to treat both models of itinerant and localized electrons simultaneously, and to discuss spin waves and their interactions in the itinerant-electron model as a direct extension of those for the Heisenberg model.[1,2] One such approximation is the random-phase approximation originally used to describe plasma oscillations[11] and more recently, to describe spin waves in the itinerant-electron model of ferromagnetism.[8,9]

In order to describe the spin-wave states, we first define the normal coordinate of a spin wave with wave vector $\mathbf{q}$, $I_{\mathbf{q}}$, such that the spin-wave state ψ_q is expressed as

$$\psi_{\mathbf{q}}=I_{\mathbf{q}}\phi_0\,, \quad (2.5)$$

where ϕ_0 is the ferromagnetic ground state with the spontaneous magnetization in the positive z axis. ($I_{\mathbf{q}}$ and $I_{\mathbf{q}}^{*}$ do not necessarily mean creation and annihilation operators of spin wave quantum. See Dyson.[1]) The state ϕ_0 in general differs from that defined previously in that each site is not necessarily occupied by an electron. Then, the spin-wave spectrum is given by the following relation, with $\hbar=1$:

$$\omega_{\mathbf{q}}I_{\mathbf{q}}\phi_0=[\mathcal{H},I_{\mathbf{q}}]\phi_0\,. \quad (2.6)$$

In the random-phase approximation (RPA) it turns out that the normal coordinate takes the following form:

$$I_{\mathbf{q}}=\textstyle\sum_{\mathbf{k}} c_{\mathbf{k}+\mathbf{q},\mathbf{k}} a_{\mathbf{k}+\mathbf{q}\uparrow}^{*} a_{\mathbf{k}\downarrow}\,. \quad (2.7)$$

Then, with the Hamiltonian (2.1), we have

$$\begin{aligned}[\mathcal{H},I_{\mathbf{q}}]=\sum_{\mathbf{k}}\{&\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k})a_{\mathbf{k}+\mathbf{q}\uparrow}^{*}a_{\mathbf{k}\downarrow}-(1/N)\sum_{\mathbf{k}_1\mathbf{k}_2}\sum_{\sigma_1}J(\mathbf{k}-\mathbf{k}_2)a_{\mathbf{k}+\mathbf{q}\uparrow}^{*}a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_2\sigma_1}a_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2\downarrow}\\ &-(U/N)\sum_{\mathbf{k}_1\mathbf{k}_2}a_{\mathbf{k}+\mathbf{q}\uparrow}^{*}a_{\mathbf{k}_1\uparrow}^{*}a_{\mathbf{k}_2\uparrow}a_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}\downarrow}-(1/N)\sum_{\mathbf{k}_1\mathbf{k}_2}\sum_{\sigma_1}J(\mathbf{k}_1-\mathbf{k}-\mathbf{q})a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_2\uparrow}^{*}a_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}-\mathbf{q}\sigma_1}a_{\mathbf{k}\downarrow}\\ &+(U/N)\sum_{\mathbf{k}_1\mathbf{k}_2}a_{\mathbf{k}_1\uparrow}^{*}a_{\mathbf{k}_2\downarrow}^{*}a_{\mathbf{k}_1-\mathbf{k}-\mathbf{q}+\mathbf{k}_2\downarrow}a_{\mathbf{k}\downarrow}\}c_{\mathbf{k}+\mathbf{q},\mathbf{k}}\,. \quad (2.8)\end{aligned}$$

In RPA, the products of four a's and a^{*}'s in the above expression are approximated in the following way:

$$\begin{aligned}a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_2\sigma_2}^{*}a_{\mathbf{k}_3\sigma_3}a_{\mathbf{k}_4\sigma_4}\cong&\langle a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_4\sigma_4}\rangle a_{\mathbf{k}_2\sigma_2}^{*}a_{\mathbf{k}_3\sigma_3}+\langle a_{\mathbf{k}_2\sigma_2}^{*}a_{\mathbf{k}_3\sigma_3}\rangle a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_4\sigma_4}\\ &-\langle a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_3\sigma_3}\rangle a_{\mathbf{k}_2\sigma_2}^{*}a_{\mathbf{k}_4\sigma_4}-\langle a_{\mathbf{k}_2\sigma_2}^{*}a_{\mathbf{k}_4\sigma_4}\rangle a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_3\sigma_3}=f_{\mathbf{k}_1\sigma_1}\{\delta_{\mathbf{k}_1\mathbf{k}_4}\delta_{\sigma_1\sigma_4}a_{\mathbf{k}_2\sigma_2}^{*}a_{\mathbf{k}_3\sigma_3}-\delta_{\mathbf{k}_1\mathbf{k}_3}\delta_{\sigma_1\sigma_3}a_{\mathbf{k}_2\sigma_2}^{*}a_{\mathbf{k}_4\sigma_4}\}\\ &+f_{\mathbf{k}_2\sigma_2}\{\delta_{\mathbf{k}_2\mathbf{k}_3}\delta_{\sigma_2\sigma_3}a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_4\sigma_4}-\delta_{\mathbf{k}_2\mathbf{k}_4}\delta_{\sigma_2\sigma_4}a_{\mathbf{k}_1\sigma_1}^{*}a_{\mathbf{k}_3\sigma_3}\}\,, \quad (2.9)\end{aligned}$$

where the angular brackets are expectation values in the ground state, and

$$f_{\mathbf{k}\sigma}\equiv\langle a_{\mathbf{k}\sigma}^{*}a_{\mathbf{k}\sigma}\rangle\,.$$

[10] See also J. C. Slater, Phys. Rev. **52**, 198 (1937).
[11] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).

Then (2.8) reduces to

$$[\mathcal{H},I_{\mathbf{q}}]_{\mathrm{RPA}}=\sum_{\mathbf{k}}\{[\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k})+2\sigma\,\tilde{u}+(1/N)\sum_{\mathbf{p},\sigma}J(\mathbf{p}+\mathbf{q},\mathbf{p})f_{\mathbf{k}-\mathbf{p},\sigma}]a_{\mathbf{k}+\mathbf{q}\uparrow}{}^{*}a_{\mathbf{k}\downarrow}$$
$$+(1/N)[\tilde{u}-J(0,\mathbf{q})](f_{\mathbf{k}+\mathbf{q}\uparrow}-f_{\mathbf{k}\downarrow})\sum_{\mathbf{k}_1}a_{\mathbf{k}_1+\mathbf{q}\uparrow}{}^{*}a_{\mathbf{k}_1\downarrow}\}c_{\mathbf{k}+\mathbf{q},\mathbf{k}}, \quad (2.10)$$

where

$$\sigma\equiv(1/2N)\sum_{\mathbf{k}}(f_{\mathbf{k}\downarrow}-f_{\mathbf{k}\uparrow}).$$

(2.6) and (2.10) determine the eigenvalue $\omega_{\mathbf{q}}$ and those coefficients $c_{\mathbf{k}+\mathbf{q},\mathbf{k}}$ for which the state $\mathbf{k}\downarrow$ is occupied and the state $\mathbf{k}+\mathbf{q}\uparrow$ is unoccupied in ϕ_0. Other c's remain undetermined by (2.6). However, for the reason given below, we determine these c's as well by imposing the following condition stronger than (2.6) with $[\mathcal{H},I_{\mathbf{q}}]$ replaced by $[\mathcal{H},I_{\mathbf{q}}]_{\mathrm{RPA}}$:

$$\omega_q I_q=[\mathcal{H},I_{\mathbf{q}}]_{\mathrm{RPA}}. \quad (2.11)$$

Substituting (2.10) into (2.11), we obtain

$$[\omega_{\mathbf{q}}-\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k})-2\sigma\,\tilde{u}-(1/N)\sum_{\mathbf{p},\sigma}J(\mathbf{p}+\mathbf{q},\mathbf{p})f_{\mathbf{k}-\mathbf{p},\sigma}]c_{\mathbf{k}+\mathbf{q},\mathbf{k}}=(1/N)[\tilde{u}-J(0,\mathbf{q})]\sum_{\mathbf{k}_1}(f_{\mathbf{k}_1+\mathbf{q}\uparrow}-f_{\mathbf{k}_1\downarrow})c_{\mathbf{k}_1+\mathbf{q},\mathbf{k}_1}. \quad (2.12)$$

From this equation we have the equation determining $\omega_{\mathbf{q}}$

$$\frac{1}{\tilde{u}-J(0,\mathbf{q})}=\frac{1}{N}\sum_{\mathbf{k}}\frac{f_{\mathbf{k}\downarrow}-f_{\mathbf{k}+\mathbf{q}\uparrow}}{\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k})+2\sigma\,\tilde{u}+(1/N)\sum_{\mathbf{p},\sigma}J(\mathbf{p}+\mathbf{q},\mathbf{p})f_{\mathbf{k}-\mathbf{p},\sigma}-\omega_{\mathbf{q}}} \quad (2.13)$$

and the expression for $c_{\mathbf{k}+\mathbf{q}\,\mathbf{k}}$

$$c_{\mathbf{k}+\mathbf{q},\mathbf{k}}=\frac{1}{\mathfrak{N}_{\mathbf{q}}{}^{1/2}}\frac{1}{2\sigma\,\tilde{u}+\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k})+(1/N)\sum_{\mathbf{p},\sigma}J(\mathbf{p}+\mathbf{q},\mathbf{p})f_{\mathbf{k}-\mathbf{p},\sigma}-\omega_{\mathbf{q}}}, \quad (2.14)$$

---

where $\mathfrak{N}_{\mathbf{q}}$ is chosen such that the normalization condition for $\psi_{\mathbf{q}}$

$$\sum_{\mathbf{k}}|c_{\mathbf{k}+\mathbf{q},\mathbf{k}}|^2 f_{\mathbf{k}\downarrow}(1-f_{\mathbf{k}+\mathbf{q}\uparrow})=1 \quad (2.15)$$

is satisfied.

For small magnitudes of wave vector $\mathbf{q}$, the smallest root of Eq. (2.13) corresponding to a spin-wave state is

$$\omega_{\mathbf{q}}=Dq^2+O(q^4), \quad (2.16)$$

where

$$D\equiv-\sigma J_{xx}(0)+\frac{1}{4\sigma N}\sum_{\mathbf{k}\sigma}f_{\mathbf{k}\sigma}\epsilon_{xx}(\mathbf{k})$$
$$-\frac{1}{4\sigma^2\,\tilde{u}}\frac{1}{N}\sum_{\mathbf{k}}(f_{\mathbf{k}\downarrow}-f_{\mathbf{k}\uparrow})\epsilon_x(\mathbf{k})^2, \quad (2.17)$$

with the abbreviations,

$$\epsilon_{\alpha\beta}\ldots(\mathbf{k})\equiv\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial k_\beta}\cdots\tilde{\epsilon}(\mathbf{k}),\quad J_{\alpha\beta}\ldots\equiv\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial k_\beta}\cdots J(\mathbf{k}), \quad (2.18)$$

$$\tilde{\epsilon}(\mathbf{k})\equiv\epsilon(\mathbf{k})+(1/N)\sum_{\mathbf{p},\sigma}J(\mathbf{p})f_{\mathbf{k}-\mathbf{p},\sigma}+\mathrm{const}. \quad (2.19)$$

$\tilde{\epsilon}(\mathbf{k})$ is taken to be a single-particle energy of an electron of either spin in the Hartree-Fock approximation.

If a band with down spin is completely filled and that with up spin empty, we can show by the periodicity of $\epsilon(\mathbf{k})$ and $J(\mathbf{p})$ that

$$D=-\sigma J_{xx}(0)-\frac{1}{4\sigma^2\,\tilde{u}}\frac{1}{N}\sum_{\mathbf{k}}\tilde{\epsilon}_x(\mathbf{k})^2. \quad (2.20)$$

III. SPIN-WAVE INTERACTIONS

We shall consider here the parts of spin-wave interactions which contribute to the temperature dependence of spin-wave energy in the first order of these interactions.[1] First, write the temperature-dependent spin-wave energy $\Omega_{\mathbf{q}}(T)$ as a sum of the temperature-independent part $\omega_{\mathbf{q}}$ obtained in the previous section and the rest denoted by $\Delta\Omega_{\mathbf{q}}(T)$:

$$\Omega_{\mathbf{q}}(T)=\omega_{\mathbf{q}}+\Delta\Omega_{\mathbf{q}}(T). \quad (3.1)$$

Considering the fact that the second term arises from the interaction of the spin wave in our attention with all other thermally excited spin waves in the first order, $\Delta\Omega_{\mathbf{q}}(T)$ can be further split up as

$$\Delta\Omega_{\mathbf{q}}(T)=\sum_{\mathbf{p}}\Delta\Omega_{\mathbf{q}\mathbf{p}}n_{\mathbf{p}}(T), \quad (3.2)$$

where $n_{\mathbf{p}}(T)$ is the average number of the spin waves of mode $\mathbf{p}$ at the temperature T. $\Delta\Omega_{\mathbf{q}\mathbf{p}}$ is the change of spin-wave energy of mode $\mathbf{q}$ due to the presence of one

spin-wave quantum of mode $\mathbf{p}$, and does no longer depend on the temperature, and is the diagonal matrix element of the interaction Hamiltonian of the two spin waves. There are other contributions to $\Delta\Omega_q(T)$, namely, those arising from higher order effects of this interaction and those involving more than two spin waves. We shall not consider these because the former would be small due to the smallness of the interaction and the latter would give rise to corrections to $\Delta\Omega_q(T)$ with higher powers of T. Thus, our task reduces to the calculation of $\Delta\Omega_{qp}$.

For this purpose, let us consider the operator,

$$[\mathcal{H},I_q]-\omega_q I_q. \tag{3.3}$$

This operator should vanish when it is applied to the ground state due to (2.6) implying that the state with one spin-wave quantum is an exact eigenstate of the total Hamiltonian. But when applied to excited states, this operator does not vanish due to the interaction between the spin wave in our attention and other excitations present. Thus, supposing that there exists an effective interaction Hamiltonian $\mathcal{H}_{\text{eff}}'$ between excitations including a spin wave of mode $\mathbf{q}$, (3.3) can be written as $\mathcal{H}_{\text{eff}}'I_q$. Since we are interested in the expectation value of $\mathcal{H}_{\text{eff}}'$ in the state with two spin waves of the mode of $\mathbf{q}$ and $\mathbf{p}$, respectively, we have

$$\langle I_p{}^*I_q{}^*\mathcal{H}_{\text{eff}}'I_qI_p\rangle=\langle I_p{}^*I_q{}^*\{[\mathcal{H},I_q)-\omega_qI_q\}I_p\rangle. \tag{3.4}$$

Use of (2.6) and introduction of a proper normalization constant finally yield

$$\Delta\Omega_{qp}=\langle I_p{}^*I_q{}^*[[\mathcal{H},I_q],I_p]\rangle/\langle I_p{}^*I_q{}^*I_qI_p\rangle. \tag{3.5}$$

Up to now in this section we made no approximation concerning the normal coordinates I_q and I_p. The knowledge of exact normal coordinates should then yield exact $\Delta\Omega_{qp}$. However, we have only approximate expressions for the normal coordinates obtained in RPA in the previous section. Thus, we used these approximate normal coordinates in (3.5) and calculated it in RPA. First, noting that the spin-wave mode is a good normal coordinate of the system, we factorize the denominator of (3.5) as

$$\langle I_p{}^*I_q{}^*I_qI_p\rangle\cong\langle I_q{}^*I_q\rangle\langle I_p{}^*I_p\rangle. \tag{3.6}$$

This quantity reduces to unity due to (2.15). Next, using (2.1) and (2.7), we obtain $[[\mathcal{H},I_q],I_p]$, which involves operators of the form $a_{k_1\uparrow}{}^*a_{k_2\uparrow}{}^*a_{k_3\downarrow}a_{k_4\downarrow}$. Thus, $\Delta\Omega_{qp}$ contains terms involving expressions such as

$$\langle a_{k_1\downarrow}{}^*a_{k_2\uparrow}a_{k_3\downarrow}{}^*a_{k_4\uparrow}a_{k_5\uparrow}{}^*a_{k_6\uparrow}{}^*a_{k_7\downarrow}a_{k_8\downarrow}\rangle.$$

Application of RPA allows us to factor this as

$$-\langle a_{k_1\downarrow}{}^*a_{k_3\downarrow}{}^*a_{k_7\downarrow}a_{k_8\downarrow}\rangle\langle a_{k_2\uparrow}a_{k_4\uparrow}a_{k_5\uparrow}{}^*a_{k_6\uparrow}{}^*\rangle.$$

This can be reduced further by using (2.9), and the final result is expressed in terms of $f_{k\sigma}$. Thus we obtain

$$\Delta\Omega_{qp}=-\frac{1}{N}\sum_{\mathbf{kk}'}\sum_{\mathbf{ll}'}c_{\mathbf{k}'+\mathbf{q},\mathbf{k}'}{}^*c_{\mathbf{l}'+\mathbf{p},\mathbf{l}'}{}^*c_{\mathbf{k}+\mathbf{q},\mathbf{k}}c_{\mathbf{l}+\mathbf{p},\mathbf{l}}(1-f_{\mathbf{k}'+\mathbf{q}\uparrow})(1-f_{\mathbf{l}'+\mathbf{p}\uparrow})f_{\mathbf{k}'\downarrow}f_{\mathbf{l}'\downarrow}\begin{Bmatrix}\mathbf{kk}' & \mathbf{ll}'\\ \mathbf{q} & \mathbf{p}\end{Bmatrix}, \tag{3.7}$$

where

$$\begin{aligned}\begin{Bmatrix}\mathbf{kk}' & \mathbf{ll}'\\ \mathbf{q} & \mathbf{p}\end{Bmatrix}\equiv&\{2J(0)+J(\mathbf{k}-\mathbf{l}-\mathbf{p})+J(\mathbf{k}-\mathbf{l}+\mathbf{q})-J(\mathbf{k}+\mathbf{q}-\mathbf{l}-\mathbf{p})-J(\mathbf{k}-\mathbf{l})\}\delta_{\mathbf{kk}'}\delta_{\mathbf{ll}'}\\&-J(-\mathbf{q})\delta_{\mathbf{k}+\mathbf{q},\mathbf{l}+\mathbf{p}}\delta_{\mathbf{ll}'}-J(-\mathbf{p})\delta_{\mathbf{k}+\mathbf{q},\mathbf{l}+\mathbf{p}}\delta_{\mathbf{kk}'}+J(\mathbf{p}-\mathbf{q},\mathbf{k}+\mathbf{q}-\mathbf{l}-\mathbf{p})\delta_{\mathbf{k}+\mathbf{q},\mathbf{l}'+\mathbf{p}}\delta_{\mathbf{k}'+\mathbf{q},\mathbf{l}+\mathbf{p}}\\&+J(\mathbf{l}-\mathbf{k}-\mathbf{q})\delta_{\mathbf{k}'+\mathbf{q},\mathbf{l}+\mathbf{p}}\delta_{\mathbf{kl}'}+J(\mathbf{k}-\mathbf{l}-\mathbf{p})\delta_{\mathbf{k}+\mathbf{q},\mathbf{l}'+\mathbf{p}}\delta_{\mathbf{k}'\mathbf{l}}-\{J(-\mathbf{q})\delta_{\mathbf{ll}'}+J(-\mathbf{p})\delta_{\mathbf{kk}'}\}\delta_{\mathbf{kl}}\\&-J(\mathbf{k}-\mathbf{l},\mathbf{q}-\mathbf{p})\delta_{\mathbf{k}'\mathbf{l}}\delta_{\mathbf{kl}'}+\{2\delta_{\mathbf{kk}'}\delta_{\mathbf{ll}'}-(\delta_{\mathbf{kk}'}+\delta_{\mathbf{ll}'})(\delta_{\mathbf{kl}}+\delta_{\mathbf{k}+\mathbf{q},\mathbf{l}+\mathbf{p}})+\delta_{\mathbf{k}'+\mathbf{q},\mathbf{l}+\mathbf{p}}\delta_{\mathbf{kl}'}+\delta_{\mathbf{k}+\mathbf{q},\mathbf{l}'+\mathbf{p}}\delta_{\mathbf{k}'\mathbf{l}}\}U.\end{aligned} \tag{3.8}$$

One can easily verify that $\Delta\Omega_{qp}$ above have two important properties:

(i) $\Delta\Omega_{qp}=\Delta\Omega_{pq}$; (3.9)

(ii) $\Delta\Omega_{qp}=0$, if either $\mathbf{q}$ or $\mathbf{p}$ vanishes. (3.10)

The first property should follow also from the fact that the interaction energy of the spin waves is written as $\frac{1}{2}\sum_{qp}\Omega_{qp}n_qn_p$. In order to verify the second property, let us assume $\mathbf{p}=0$. Then, from (2.14), $c_{l+p,l}$ is seen to be independent of $\mathbf{l}$, and (3.7) involves the following sum:

$$\sum_{\mathbf{l}}\begin{Bmatrix}\mathbf{kk}' & \mathbf{ll}'\\ \mathbf{q} & 0\end{Bmatrix},$$

which is shown to vanish by using (3.8). The second property implies that the zero-wave vector spin wave has no interaction with other spin waves as one expects. In the previous section, we mentioned about a certain ambiguity in determining $c_{k+q,k}$. It is this property (ii) which made us impose the stronger condition (2.11) rather than (2.6) to eliminate this ambiguity. Other choices, for instance, where one assumes that all $c_{k+q,k}$ other than those for which the state $\mathbf{k}+\mathbf{q}\uparrow$ is empty and the state $\mathbf{k}\downarrow$ is occupied in the ground state, are set equal to zero, do not have this property.

Here it is appropriate to remark a little more on the nature of this ambiguity. First, note that if we have the exact normal coordinates, the exact spin-wave spectrum,

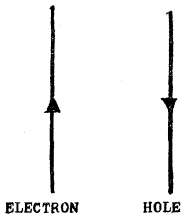

FIG. 1. Diagrams of an electron and a hole.

and the exact ground-state wave function, (3.5) can be rewritten as (choosing $\langle \mathcal{H} \rangle = 0$)

$$\Delta\Omega_{\mathbf{qp}} = \langle I_{\mathbf{q}}^* I_{\mathbf{p}}^* \mathcal{H} I_{\mathbf{p}} I_{\mathbf{q}} \rangle / \langle I_{\mathbf{q}}^* I_{\mathbf{p}}^* I_{\mathbf{p}} I_{\mathbf{q}} \rangle - \omega_q - \omega_p .$$

Thus, due to the commutativity of different I's, the ambiguity should have no effect on $\Delta\Omega_{\mathbf{qp}}$ as long as one performs exact calculations. Thus, the ambiguity matters only when one introduces approximations, and such ambiguity should be eliminated at each step of the approximation in a way consistent with other requirements such as (3.10), as we have done here.

Another important consequence of (3.7) is that in the limit of localized spins, our $\Delta\Omega_{\mathbf{qp}}$ reduces to the correct result of Dyson and others.[1,2] In order to see this, we drop the kinetic energy in the Hamiltonian and suppose that in the ground state, the band with down spin is completely filled, and that with up spin empty. Then, from (2.14) and (2.15), $c_{\mathbf{k+q},\mathbf{k}}$ and $c_{\mathbf{l+p},\mathbf{l}}$ become independent of $\mathbf{k}$ and $\mathbf{l}$, respectively, and are equal to $N^{-1/2}$. (3.7) then becomes

$$\Delta\Omega_{\mathbf{qp}} = -N^{-3} \sum_{\mathbf{kk'}} \sum_{\mathbf{ll'}} \begin{Bmatrix} \mathbf{kk'} & \mathbf{ll'} \\ \mathbf{q} & \mathbf{p} \end{Bmatrix}, \qquad (3.11)$$

where the summations are over the first Brillouin zone of the reciprocal lattice space. When one makes use of the fact that $\sum_{\mathbf{k}} J(\mathbf{k}) = N J_{\mathbf{ll}} = 0$ and that $J(\mathbf{k})$ is periodic in the reciprocal lattice space, (3.11) reduces to

$$\Delta\Omega_{\mathbf{qp}} = -(2/N)[J(0) - J(\mathbf{q}) - J(\mathbf{p}) + J(\mathbf{q}-\mathbf{p})]. \qquad (3.12)$$

This agrees with the result of the Heisenberg model.[1,2] Therefore, our result (3.7) may be regarded as a direct extension of the result of Dyson and others[1,2] to the case of itinerant electron model of ferromagnetism.

Before proceeding to the consequences of our result (3.7), it is helpful to give its diagrammatical interpretation. We take the ferromagnetic ground state as the vacuum and represent an electron outside the Fermi spheres and a hole inside by a vertical line with an arrow directed up and down, respectively (Fig. 1). Then a spin-wave quantum is represented by two vertical lines corresponding to an electron with up spin and momentum $\mathbf{k}+\mathbf{q}$ and a hole with down spin and momentum $\mathbf{k}$ (Fig. 2). The hole is labeled by the spin and momentum of the electron annihilated to create the hole.

The interactions among electrons and holes are given by the last two terms of (2.1). The form of the normal coordinate (2.7) tells us that in spin-wave states we need to consider only electrons with up spin and outside the Fermi sufrace and holes with down spin and inside the Fermi surface. Then the last two terms of (2.1) consist of three types of interactions:

(a) Interactions between two electrons or two holes of the same spin described by $J(\mathbf{k})$ [Fig. 3(a)].

(b) Interactions accompanying annihilation and creation of electron-hole pairs described by $J(\mathbf{k})$ [Fig. 3(b)].

(c) Interactions between an electron and a hole with opposite spins described by U [Fig. 3(c)].

The spin-wave interactions are constructed in the following way: First, draw four lines representing two spin waves. Insert one interaction of the kind (a), (b), or (c) described above in all possible ways among these four lines, excluding those connecting two lines which belong to the same spin wave because these are already taken into account in describing each noninteracting single spin waves. Taking into account the possibility of electron (hole) exchanges in the final states, each interaction of the types (a), (b), and (c) gives rise to eight different diagrams of spin-wave interactions. Thus, we have altogether 24 different diagrams which precisely correspond to 24 terms (counting multiplicity) of (3.8). Some typical diagrams arising from each of the types (a), (b), and (c) (above) are shown in Fig. 4(a), (b), and (c), respectively.

The first 16 diagrams or terms belonging to the types of Fig. 4(a) and (b) are further divided into two groups of eight diagrams each. The first group consists of those diagrams [or terms of (3.8)] in which arguments of J involve only $\mathbf{q}$ and $\mathbf{p}$, whereas the second group consists of those in which arguments of J cannot be written solely in terms of $\mathbf{q}$ and $\mathbf{p}$. The terms of (3.8) corresponding to the first group do not vanish in the limit of localized electrons and give rise to Dyson's result (3.12). The terms corresponding to the second group vanish in this limit. One can show that the diagrams of the second group are derived from those of the first by exchanging electrons (holes) of the same kind among themselves in the final state in such a way that no diagrams appear more than once.

Here a remark is in order. From (3.7) there is a restriction on possible final electronic states, namely that the states $\mathbf{k}'+\mathbf{q}_\uparrow$ and $\mathbf{l}'+\mathbf{p}_\uparrow$ should be outside the Fermi surface, whereas the states $\mathbf{k}'_\downarrow$ and $\mathbf{l}'_\downarrow$ should be inside. There is no such restriction on the initial electronic states. Thus, in order to make our diagrammatic inter-

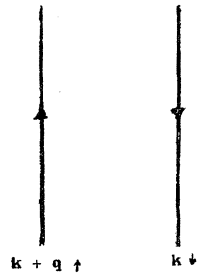

FIG. 2. A diagram of a spin wave.

pretation more precise we must include diagrams in which initial electronic states do not satisfy the corresponding restriction. Thus, for instance, in the diagram of Fig. 4(b), we must allow such situations in which the state $\mathbf{k}_\downarrow$ is outside the Fermi surface and the state $\mathbf{l}+\mathbf{p}_\uparrow$ inside. The lines representing these states then are suppressed in the diagram and this situation is represented in Fig. 5, where the cross indicates the interaction of created electron-hole pair with other ground-state electrons.

IV. SPIN-WAVE INTERACTIONS

Having obtained an expression for spin-wave interaction (3.7) and (3.8) and some of its properties, we proceed to investigate it for small wave vectors of the spin waves involved. For simplicity we assume that the band with up spin is empty in the ground state, that is, $f_{\mathbf{k}\uparrow}=0$.

In (3.7) we can drop all f's if the summations over $\mathbf{k}'$ and $\mathbf{l}'$ are restricted within the Fermi sphere of down-spin band. We first perform summations over $\mathbf{k}$ snd $\mathbf{l}$ thus eliminating all the Kronecker's delta in (3.8), and then expand the resulting expression in powers of $\mathbf{q}$ and $\mathbf{p}$, where no singularity appears in the procedure. Therefore, if one takes into consideration the two properties of $\Delta\Omega_{\mathbf{qp}}$, (i) and (ii) in III, $\Delta\Omega_{\mathbf{qp}}$ has the following general form for small values of q and p:

$$\Delta\Omega_{\mathbf{qp}}=-C_1\mathbf{q}\cdot\mathbf{p}-C_2(\mathbf{q}\cdot\mathbf{p})^2 -C_3q^2p^2-C_4(\mathbf{q}\cdot\mathbf{p})(q^2+p^2). \quad (4.1)$$

Here, the first and the last terms give rise to scattering between spin waves, but do not contribute to $\Delta\Omega_{\mathbf{q}}(T)$ because they vanish when averaged over all directions of p. The second and the third terms give spin-wave interactions of the form q^2p^2 after the same averaging. This is in agreement with Marshall's argument[6] on general considerations. In the following we drop the last term in (4.1) out of consideration.

Evaluation of the coefficients C_1, C_2, and C_3 is elementary but tedious. For C_1, we obtain

$$C_1=-\frac{2}{3N}\sum_\alpha\left\{\bar{J}_{\alpha\alpha}+\frac{1}{\sigma^2\,\tilde{u}N_e}\sum_{\mathbf{k}}{}'\,\epsilon_\alpha(\mathbf{k})^2\right\},\ (\alpha=x,y,z) \quad (4.2)$$

where $\sum'$ means the summation within the Fermi sphere

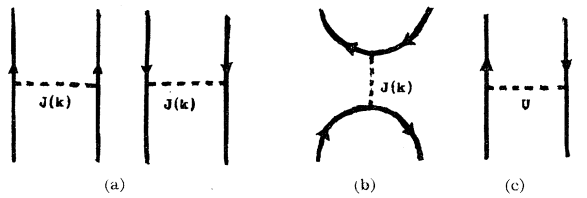

FIG. 3. (a) Diagrams of interactions between two electrons or two holes of the same spin. (b) A diagram of an interaction accompanying annihilation and creation of electron-hole pairs. (c) A diagram of an interaction between an electron and a hole with opposite spins.

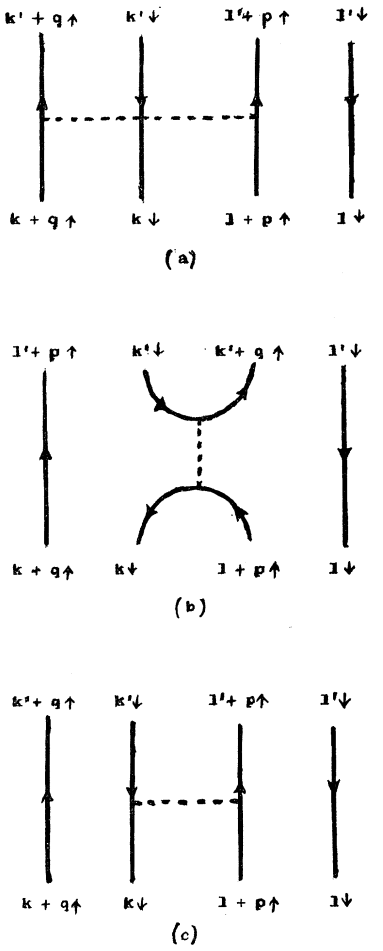

FIG. 4. (a), (b), and (c) represent typical diagrams of spin-wave interactions involving interactions of Fig. 3(a), 3(b), and 3(c), respectively.

and we also define

$$\bar{J}_{\alpha\beta}\ldots\equiv J_{\alpha\beta}\ldots(0)-\frac{1}{N_e{}^2}\sum_{\mathbf{k}}{}'\sum_{\mathbf{l}}{}'\,J_{\alpha\beta}\ldots(\mathbf{k}-\mathbf{l}). \quad (4.3)$$

For the Heisenberg model, (3.12) yields,

$$C_1=-(2/3N)\textstyle\sum_\alpha J_{\alpha\alpha}(0). \quad (4.4)$$

(4.2) differs from (4.4) in two respects:

(i) Appearance of the additional term due to electron kinetic energy and Coulomb repulsion.

(ii) The modification of the exchange interaction term.

These same features also appear in C_2 and C_3, and will be discussed later. Let us now rewrite (2.17) as follows:

$$D=-\frac{N_e}{2N}\bar{J}_{xx}+\frac{1}{2N_e}\sum_{\mathbf{k}}{}'\frac{\partial^2}{\partial k_x{}^2}\epsilon(\mathbf{k})-\frac{1}{4N\sigma^2\,\tilde{u}}\sum_{\mathbf{k}}{}'\,\epsilon_x(\mathbf{k})^2. \quad (2.17')$$

Using this expression, (4.2) can be also written as, noting

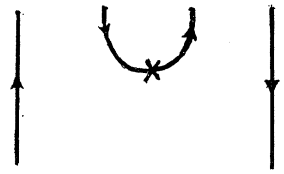

FIG. 5. Special case of Fig. 4(b), where the state $\mathbf{k}\downarrow$ is occupied and the state $\mathbf{l}+\mathbf{p}\uparrow$ is empty in the spin-wave state.

the cubic symmetry of the crystal,

$$C_1\equiv\left\{\frac{2}{S_{\rm tot}}D-\frac{2}{N_e{}^2}\sum_{\mathbf{k}}{}'\frac{\partial^2}{\partial k_x{}^2}\epsilon(\mathbf{k})\right\}\mathbf{q}\cdot\mathbf{p}, \quad (4.10)$$

where $S_{\rm tot}=N\sigma=N_e/2$ is the magnitude of the total spin of the system. If the second term of the expression inside the curly bracket above is absent, the relation reduces to that obtained by Keffer and Loudon[2] for the Heisenberg model and called the structure-independent spin-wave interaction. The second term vanishes if there is no kinetic energy or if the band with down spin is completely filled.

We now turn to C_2 and C_3. In order to simplify calculations, we shall consider two simple cases of our model: Case (i). The exchange interaction is absent. In this case the spin-wave interaction arises from the processes such as given in Fig. 4(c) and we get

$$C_2=-\frac{1}{2N\sigma mk_F{}^2}\left[\frac{2}{5}\xi^2+\frac{66}{175}\xi^3\right], \quad (4.11a)$$

$$C_3=\frac{1}{2N\sigma mk_F{}^2}\left[\frac{2}{5}\xi^2+\frac{6}{35}\xi^3\right], \quad (4.11b)$$

where

$$\xi\equiv\frac{k_F{}^2}{2m\sigma U}=\frac{\epsilon_F}{\sigma U}, \quad (\epsilon_F\cdot\text{Fermi energy}). \quad (4.12)$$

[T. Izuyama used an expression for $\Delta\Omega_{\mathbf{qp}}$ slightly different from (3.5) to calculate C_2 and C_3 and obtained a similar result (private communication). Also see Ref. 9.] case (ii). The kinetic energy is negligibly small. Here, the itinerant character of electrons is reflected in the fact that the band is only partially filled with electrons. In this case, we obtain the following contribution to $\Delta\Omega_{\mathbf{qp}}$ from C_2 and C_3 terms of (4.1):

$$-\frac{1}{2N}\sum_{\alpha\beta\gamma\delta}\bar{J}_{\alpha\beta\gamma\delta}q_\alpha q_\beta p_\gamma p_\delta \quad (\alpha,\beta,\gamma,\delta=x,y,z). \quad (4.13)$$

This is to be compared with the corresponding expression for the Heisenberg model, which is, from (3.12),

$$-\frac{1}{2N}\sum_{\alpha\beta\gamma\delta}J_{\alpha\beta\gamma\delta}(0)q_\alpha q_\beta p_\gamma p_\delta. \quad (4.14)$$

We thus see these two cases represent, respectively, two new features in C_1 of the itinerant-electron model mentioned previously. If we have holes instead of electrons, that is, the band with the down spin is almost full, we have to change the sign of m in (4.11). The two terms of $\bar{J}_{\alpha\beta\gamma\delta}$ in (4.13) [see (4.3)] arise from those diagrams belonging to the first and second groups of III, respectively. Thus, the modification of J's in (4.13) is due to the electron (hole) exchange effect, which cancels in the limit of localized electrons (the full band).

In the general case, we also get terms additional to (4.11) and (4.13) which involve both the kinetic energy and the exchange interaction.

Let us now examine the magnitude of $\Delta\Omega_{qp}$ in case (i) which we denote as $\Delta\Omega_{qp}{}^K$ relative to that of the Heisenberg model (4.14), which we denote as $\Delta\Omega_{qp}{}^H$. Averaging over all directions of $\mathbf{p}$ yields

$$\langle\Delta\Omega_{qp}{}^K\rangle=-\frac{\epsilon_F}{15N\sigma^3m^2U^2}\left\{1+\frac{6}{35}\xi\right\}q^2p^2, \quad (4.15)$$

where $\epsilon_F=k_F{}^2/2m$. Assuming the nearest-neighbor exchange interaction for the Heisenberg model, and noting $J_{xxxx}(0)\sim J_{xxyy}(0)\sim a^4Jz$, where J is the magnitude of the exchange interaction and z the number of nearest neighbors, we obtain after averaging over the directions of $\mathbf{p}$,

$$\langle\Delta\Omega_{qp}{}^H\rangle\cong-(a^4/2N)zJq^2p^2. \quad (4.16)$$

The ratio of (4.15) and (4.16) is

$$|\langle\Delta\Omega_{\mathbf{qp}}{}^K\rangle/\langle\Delta\Omega_{\mathbf{qp}}{}^H\rangle|=\frac{16}{15\sigma^2z}\frac{\epsilon_F}{D/a^2}\left(\frac{W}{U}\right)^2\left(1+\frac{6}{35}\xi\right), \quad (4.17)$$

where $D=2\sigma a^2J$ and has the same meaning as in (2.16) for the Heisenberg model and $W=1/2ma^2$. In the case of Ni, $D=0.47\times10^{-28}$ erg cm^2,[4] $a=3.52$ Å, $m=28$ times electron mass,[9] $z=12$, $\sigma=0.3$ corresponding to 0.6 hole per atom and $k_F=1.49$ Å^{-1}, and the above ratio becomes $1.5\times10^{-3}[1+(6/35)\xi]/U^2$, where U is expressed in electron volts and is typically about 10 eV. (If we consider the effect of electron correlations, U might be taken to be of the order of $\epsilon_F=0.3$ eV in our estimation.) Although this estimation is very crude, it is unlikely that $\Delta\Omega_{qp}{}^K$ can explain the rather large spin-wave interactions observed in Ni and permalloy. Moreover, when we are dealing with holes rather than electrons, we get positive $\langle\Delta\Omega_{qp}{}^K\rangle$.

We now turn to the case (ii). Writing $J(\mathbf{k})$ as

$$J(\mathbf{k})=\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{R}}J_{\mathbf{R}}, \quad (4.18)$$

where $J_{\mathbf{R}}$ is the exchange interaction constant of electrons separated by $\mathbf{R}$, we obtain from (4.3)

$$\bar{J}_{\alpha\beta\gamma\delta}=\sum_{\mathbf{R}}R_\alpha R_\beta R_\gamma R_\delta\left(1-\frac{1}{N_e{}^2}\sum_{\mathbf{k},\mathbf{l}}{}'e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{R}}\right)J_{\mathbf{R}}. \quad (4.19)$$

Thus, it looks as if, in C_1, (4.2) as well as in (4.13) and D, (2.17′), the exchange interaction $J_{\mathbf{R}}$ of the localized

electron model, is replaced by $\tilde{J}_{\mathbf{R}}$, given as

$$\tilde{J}_{\mathbf{R}}\equiv\left(1-\frac{1}{N_e{}^2}\sum_{\mathbf{k},\mathbf{l}}{}' e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{R}}\right)J_{\mathbf{R}}. \tag{4.20}$$

When the down-spin band is slightly filled, assuming a parabolic electron band of spherical symmetry, we can carry out the summations in (4.20) and obtain

$$\tilde{J}_{\mathbf{R}}=g_e(k_F R)J_{\mathbf{R}}, \tag{4.21}$$

where

$$g_e(x)\equiv 1-\left\{\frac{3(x\cos x-\sin x)}{x^3}\right\}^2=x^2/5, \quad x\ll 1$$
$$=1, \qquad x\gg 1. \tag{4.22}$$

The function $g_e(x)$ starts with zero at $x=0$ and rapidly approaches unity as x increases beyond 1. Thus, in the modified exchange interaction, the short-range part of the original exchange interaction is suppressed, whereas the long-range part remains unaffected. This is natural since all the electrons available have wave numbers smaller than k_F, and hence cannot distinguish the distance smaller than $\sim k_F{}^{-1}$.

If the band with down spin is almost full, assuming a parabolic hole band of spherical symmetry, we obtain an expression similar to (4.21).

$$\tilde{J}_{\mathbf{R}}=g_h(k_F R)J_{\mathbf{R}}, \tag{4.23}$$

where

$$g_h(x)\equiv 1-\left(\frac{N-N_e}{N_e}\right)^2\left\{\frac{3(x\cos x-\sin x)}{x^3}\right\}^2 \tag{4.24}$$

and k_F is the Fermi momentum of holes. When $N-N_e \ll N_e$, the short-range part of $\tilde{J}_{\mathbf{R}}$ is only slightly suppressed and the long-range part is not affected.

Thus, we see that if there exists in the original exchange interaction between localized electrons a long-range part, such as suggested by Roth,[12] it will certainly remain in the itinerant-electron model.

V. CONCLUSION AND SOME REMARKS

In the preceding sections, we have been able to deduce spin-wave interactions on the basis of itinerant-electron model of ferromagnetism. For the simple case discussed in IV, the form of spin-wave interactions turned out to be the same as in the Heisenberg model. This gives rise to the temperature dependence of the spin-wave frequency proportional to $T^{5/2}$. However, it has not been possible to attribute the observed large spin-wave interactions to the itinerant character of magnetic electrons.

In this work, for the sake of simplicity, we restricted ourselves to the simple model of ferromagnetism. It would be interesting to apply our method to more realistic models. In particular, one can take into account the degeneracy of the band, which seems to be important in the problem of metallic ferromagnetism. Furthermore, one may also develop a method of incorporating the effects of higher random phase approximation[13] in the present theory.

On the other hand, there is another contribution to the temperature dependence of spin-wave spectrum which has not been considered in this paper. That is, the effect due to the smearing out with temperature of the Fermi distribution function of electrons, which varies as T^2, and thus should be more important than spin-wave interactions at lower temperatures.[9] This can be obtained from (2.17), where $f_{\mathbf{k}\sigma}$ now depends on the temperature. Crude estimation shows that in the temperature range in which experiments have been performed so far, T^2 term is rather small compared to the $T^{5/2}$ term. [It was pointed out that if the exchange interaction between magnetic electrons is due to the conduction electron (Ruderman-Kittel-Yosida interaction), additional T^2 term would appear due to the smearing out of the Fermi distribution function of conduction electrons (Ref. 9).] So far there is no experimental indication of the existence of T^2 term.

Finally, we make a few remarks on the recent work of Nakamura.[14] In his theory, the spin-wave interaction consists of two parts, Γ_1 and Γ_2 in his notation. He mentions that each part does not vanish when one of the interacting spin wave has a vanishing wave number. On the other hand, it turns out that Γ_2 coincides with our $\Delta\Omega_{\mathbf{q}\mathrm{p}}$ if the normal coordinates used are the same. Thus, it appears from our result that the normal coordinates he used are different from ours. Note the ambiguity we mentioned before. Γ_1 part is more difficult to understand. If it is true, it should come in our theory from higher random-phase approximation. However, we point out that, as we have done in our theory, it is possible to perform the calculation in such a way that at each stage of the approximation the spin-wave interaction vanishes when the wave number of one of interacting spin waves vanishes. This is guaranteed by the fact that the normal coordinate of the zero wave vector spin wave,

$$I_0=\text{const}\sum_{\mathbf{k}} a_{\mathbf{k}\uparrow}{}^* a_{\mathbf{k}\downarrow},$$

is exact and thus commutes with the total Hamiltonian, and the normal coordinates commute with each other. We feel that this way of calculation is safer than invoking cancellation of different contributions when one stops at a certain stage of the approximation.

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APPENDIX

Here we discuss the Green's function approach to the problem of spin waves in the itinerant-electron model of ferromagnetism. We adopt the causal Green's function which does not involve temperature defined as follows[15]: For any operators A and B,

$$\langle\langle A;B\rangle\rangle(t)\equiv -i\langle TA(t)B\rangle, \tag{A1}$$

where T denotes a well-known time-ordering operator[15] and $\langle\ \rangle$ denotes the expectation value in the ground state.

First, we discuss a single spin wave, which has already been treated by Kubo *et al.*[9] Introducing the following Green's function,

$$G_{\mathbf{k}\mathbf{k}';\mathbf{q}}(t)\equiv\langle\langle a_{\mathbf{k}\downarrow}{}^{*}a_{\mathbf{k}+\mathbf{q}\uparrow};\, a_{\mathbf{k}'+\mathbf{q}\uparrow}{}^{*}a_{\mathbf{k}'\downarrow}\rangle\rangle(t), \tag{A2}$$

its Fourier transform with respect to time (for which we use the same notation) satisfies

$$\omega G_{\mathbf{k}\mathbf{k}';\mathbf{q}}=\delta_{\mathbf{k}\mathbf{k}'}/(2\pi)(f_{\mathbf{k}\downarrow}-f_{\mathbf{k}+\mathbf{q}\uparrow}) +\langle\langle[a_{\mathbf{k}\downarrow}{}^{*}a_{\mathbf{k}+\mathbf{q}\uparrow},\mathcal{H}];\, a_{\mathbf{k}'+\mathbf{q}\uparrow}{}^{*}a_{\mathbf{k}'\downarrow}\rangle\rangle. \tag{A3}$$

The first decoupling approximation for the second term of this expression coincides with the result of RPA in the text, in II and yields a linear closed equation for $G_{\mathbf{k}\mathbf{k}';\mathbf{q}}$, which can be solved by introducing another Green's function as follows,

$$G_{\mathbf{q};\mathbf{k}'}\equiv\sum_{\mathbf{k}} c_{\mathbf{k}+\mathbf{q},\mathbf{k}}{}^{*}G_{\mathbf{k}\mathbf{k}';\mathbf{q}}, \tag{A4}$$

where $c_{\mathbf{k}+\mathbf{q},\mathbf{k}}$ satisfies (2.12) and $\omega_{\mathbf{q}}$ is given by (2.13). Thus, we obtain

$$(\omega-\omega_{\mathbf{q}})G_{\mathbf{q};\mathbf{k}'}=(1/2\pi)(f_{\mathbf{k}'\downarrow}-f_{\mathbf{k}'+\mathbf{q}\uparrow})c_{\mathbf{k}'+\mathbf{q},\mathbf{k}'}{}^{*}. \tag{A5}$$

Here, the ambiguity in determining c's mentioned in the text does not arise. This is, however, not taken to be the superiority of the Green's function approach, but rather, this is because the Green's function approach requires more than necessary for determining the normal coordinates of spin waves.

Now, we turn to the problem of spin-wave interactions. This has also been treated by Izuyama and Kubo[9] in a somewhat indirect manner, in which they obtain temperature-dependent spin-wave frequency by making use of one-magnon temperature Green's function. Here, we shall start with the two-magnon Green's function, and obtain spin-wave interaction directly. Thus, supposing that the exact normal coordinates $I_{\mathbf{q}}$ of the spin waves are known, we define the two-magnon Green's function by

$$G_{\mathbf{q};\mathbf{p}}\equiv\langle\langle I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*};\, I_{\mathbf{p}}I_{\mathbf{q}}\rangle\rangle. \tag{A6}$$

This, then, satisfies the equation

$$\omega G_{\mathbf{q};\mathbf{p}}=(1/2\pi)\langle[I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*},I_{\mathbf{q}}I_{\mathbf{p}}]\rangle +[\omega_{\mathbf{q}}+\omega_{\mathbf{p}}+\textstyle\sum_{\mathbf{q}\mathbf{p}}(\omega)]G_{\mathbf{q};\mathbf{p}}, \tag{A7}$$

where $\omega_{\mathbf{q}}$ and $\omega_{\mathbf{p}}$ are exact spin-wave frequencies with wave vectors $\mathbf{q}$ and $\mathbf{p}$, respectively, supposedly known, and $\sum_{\mathbf{q}\mathbf{p}}(\omega)$ is the mass operator defined by

$$\textstyle\sum_{\mathbf{q}\mathbf{p}}(\omega)\equiv\langle\langle[I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*},\mathcal{H}]';\, I_{\mathbf{q}}I_{\mathbf{p}}\rangle\rangle/\langle\langle I_{\mathbf{q}}{}^{*}I_{\sigma}{}^{*};\, I_{\mathbf{q}}I_{\mathbf{p}}\rangle\rangle, \tag{A8}$$

where

$$[I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*},\mathcal{H}]'\equiv[I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*}\mathcal{H}]-(\omega_{\mathbf{q}}+\omega_{\mathbf{p}})I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*}. \tag{A9}$$

If one could evaluate $\sum_{\mathbf{q}\mathbf{p}}(\omega)$ directly for $\omega=\omega_{\mathbf{q}}+\omega_{\mathbf{p}}$, this would give the interaction energy between spin waves $\mathbf{q}$ and $\mathbf{p}$. This is made difficult especially because of the lack of the exact expressions for I's, ω's, and the ground state. Here, we shall restrict ourselves to the simple case when the down-spin band is partially filled and the up-spin band empty, and show that in this simple case $\sum_{\mathbf{q}\mathbf{p}}(\omega)$ reduces to (3.5) of the text for weak spin-wave interaction. In this case we can neglect spin-wave interactions not involved in $[I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*},\mathcal{H}]$ and thus, we obtain

$$\begin{aligned}&\langle\langle I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*};\, I_{\mathbf{p}}I_{\mathbf{q}}\rangle\rangle(t)\\ &\quad=-i\langle T(I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*}I_{\mathbf{p}}(-t)I_{\mathbf{q}}(-t))\rangle\\ &\quad=-i[\langle I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*}I_{\mathbf{p}}I_{\mathbf{q}}\rangle\theta(t)+\langle I_{\mathbf{p}}I_{\mathbf{q}}I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*}\rangle\theta(-t)]\\ &\qquad\times\exp[-i(\omega_{\mathbf{q}}+\omega_{\mathbf{p}})t],\end{aligned} \tag{A10}$$

where

$$\begin{aligned}\theta(t)&=1 \quad \text{for} \quad t>0\\ &=0 \quad \text{for} \quad t<0.\end{aligned} \tag{A11}$$

Noting that in this simple case $\langle I_{\mathbf{p}}I_{\mathbf{q}}I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*}\rangle=0$, the Fourier transformed Green's function then becomes

$$\begin{aligned}&\langle\langle I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*};\, I_{\mathbf{q}}I_{\mathbf{p}}\rangle\rangle(\omega)\\ &\quad\cong-(i/2\pi)\langle I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*}I_{\mathbf{p}}I_{\mathbf{q}}\rangle/[\epsilon+i(\omega-\omega_{\mathbf{q}}-\omega_{\mathbf{p}})],\end{aligned} \tag{A12}$$

where ϵ is a small positive number. In the same way we also obtain

$$\begin{aligned}&\langle\langle[I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*},\mathcal{H}]';\, I_{\mathbf{p}}I_{\mathbf{q}}\rangle\rangle(\omega)\\ &\qquad\cong-(i/2\pi)\langle[I_{\mathbf{q}}{}^{*}I_{\mathbf{p}}{}^{*},\mathcal{H}]'I_{\mathbf{p}}I_{\mathbf{q}}\rangle/\\ &\qquad\qquad[\epsilon+i(\omega-\omega_{\mathbf{q}}-\omega_{\mathbf{p}})].\end{aligned} \tag{A13}$$

Substituting (A12) and (A13) into (A8), and noting that the ground state, the spin-wave frequencies, and the normal coordinates are supposed to be exact, we easily see that $\sum_{\mathbf{q}\mathbf{p}}(\omega)$ reduces to $\Delta\Omega_{\mathbf{q}\mathbf{p}}$ of (3.5). For a more general case, this correspondence is not so manifest, but we feel our expression (3.5) is sufficiently clear.

[15] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).